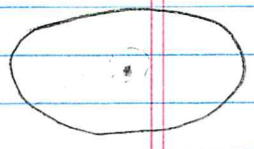


Example: Let  $x \in M$  and  $D \ni x$  be a complex coordinate chart in which  $x=0 \in D \cong D_1 \subset \mathbb{C}$ . Define the "point bundle"  $\mathcal{O}(x)$  to be the hol. line bundle with

$$U^1 = D \cong U^2 = M \setminus x$$

$$u^1_2 = z$$



Note:  $\mathcal{O}(x)$  has a global section

$$s = \begin{cases} 1 & \text{on } M \setminus x \\ z & \text{on } D \end{cases}$$

$$u^1_2 \cdot 1 = z \quad \checkmark$$

Let  $\varphi \in C^\infty(M)$  with  $\varphi = \begin{cases} 0 & \text{on } D_{1/2} \\ 1 & \text{on } M \setminus D \end{cases}$

Define  $A = -\varphi dz z^{-1}$  on  $D$ .  
 $\Rightarrow \nabla_A s = dz - \varphi dz \cdot z^{-1} \cdot z = dz(1-\varphi) = 0$  on  $M \setminus D$ .  
 $F_A = dA = -d(\varphi d \log z)$ ,  $\text{supp } |F_A| \subset \overline{D} \setminus D_{1/2}$

$$\text{degree}(E) := \int_M c_1(E) = -\frac{i}{2\pi} \int_D d(\varphi d \log z)$$

$$= -\frac{i}{2\pi} \int_{S_1} d \log z = \frac{-i \cdot 2\pi i}{2\pi} = 1$$



[Con: Not all bundles are flat!]

Note:  $T(U, \mathcal{O}(x)) \simeq \{ \text{meromorphic forms on } U \text{ w/ single pole at } x \}$

Recall: Every hol. v.b.  $E$  w/ metric  $\langle \cdot, \cdot \rangle$  has a unique compatible (Chern) connection  $A$  with  $D_A^{0,1} = \bar{\partial}_E$ .

Conversely: Theorem: Let  $A$  be a unitary connection on a v.b.  $E \rightarrow M$ , with  $M$  a Riemann surface ( $\dim_{\mathbb{R}} M = 1$ ). Then  $\bar{\partial}_A = D_A^{0,1}$  defines a hol. structure  $\bar{\partial}$  on  $E$ .

Proof: Let  $x_0 \in D \subset M$  and choose a local unitary frame  $\{e_\alpha\}$  over  $D$ . Write

$$\bar{\partial}_A e_\alpha = -a^\beta_\alpha d\bar{z} \otimes e_\beta \dots$$

We want to show the existence of a local holomorphic frame  $\{f_\mu\}$  with  $\bar{\partial}_A f_\mu = 0$ .

$$\text{Writing } f_\mu = g^\alpha_\mu e_\alpha \Rightarrow \bar{\partial}_A f_\mu = \left( \frac{\partial}{\partial \bar{z}} g^\alpha_\mu + a^\beta_\alpha g^\alpha_\mu \right) d\bar{z} \otimes e_\beta$$

Need to solve:  $(*) \quad \frac{\partial g}{\partial \bar{z}} + a \cdot g = 0, \quad g \in GL(n, \mathbb{C}).$

Note: since  $x_0 \in M$  was arbitrary, it is sufficient to solve  $(*)$  in a smaller disk  $D_r$ . Rescaling by

$$\begin{aligned} D &\rightarrow D_r \\ z &\mapsto r z \\ \Rightarrow \quad \frac{\partial}{\partial \bar{z}} &\mapsto \frac{1}{r} \frac{\partial}{\partial \bar{z}} \\ a(z) &\mapsto a(r \cdot z). \end{aligned}$$

$$(*) : \quad \frac{1}{r} \frac{\partial}{\partial \bar{z}} g + a(r, z) \cdot g = 0, \quad g \in GL(n, \mathbb{C})$$

$$(\Rightarrow) \quad \frac{\partial}{\partial \bar{z}} g + r a(r, z) \cdot g = 0$$

$\Rightarrow$  By choosing  $r$  small, we may assume that  $\sup |a| < N$

for any  $N > 0$ . [still working on  $D = D_1$ ]

We may also choose a smooth cut off



and multiply  $a \mapsto \varphi \cdot a$ . [since we need only solve near  $\gamma_0$ !]

Hence, wlog, we can assume:

$$(\ast) \quad \text{supp } |a| \subset D, \quad \sup |a| < N.$$

Recall the Cauchy kernel  $\bar{\partial} \left( -\frac{1}{2\pi i z} \right) = \delta_0$ .

For any  $\varphi$ thly supported form  $\Theta(z)$  on  $\mathbb{C}$ , let

$$(L\Theta)(w) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\Theta(z)}{z-w} \left( \frac{i dz d\bar{z}}{z} \right)$$

$$\Rightarrow \frac{\partial}{\partial \bar{z}} L\Theta = \Theta.$$

To solve  $(\ast)$ , write  $g = 1 + f$

$$\Rightarrow \frac{\partial f}{\partial \bar{z}} = -a \cdot (1 + f)$$

Since  $a$  has cpt support, suffices to solve

$$f = L(-a \cdot (1+f))$$

Estimate on  $L$ :

$$\int_D \left| \frac{1}{z} \right| \leq \int_D \frac{1}{r} \cdot 2\pi r dr = 2\pi$$

$$\text{Supp } a \subset D \Rightarrow \|L(a \cdot \theta)\|_{L^\infty(C)} \leq \|a \cdot \theta\|_{L^\infty(C)} \cdot \frac{2\pi}{2\pi} \leq N \cdot \|\theta\|_{L^\infty}$$

$\Rightarrow$  for  $N < 1$ ,  $T(f) := L(-a \cdot (1+f))$

is a contraction mapping! [L is linear]

$$T: L^\infty(C) \rightarrow L^\infty(C)$$

$\|a\| < N$   
 $f=0$  initially.

$\Rightarrow \exists!$  fixed-point  $f \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} T^n(a) = T(f)$

Note also that  $\|f\|_{L^\infty} \leq \frac{N + N^2 + N^3 + \dots}{1-N}$  [contraction mapping]  
Careful.

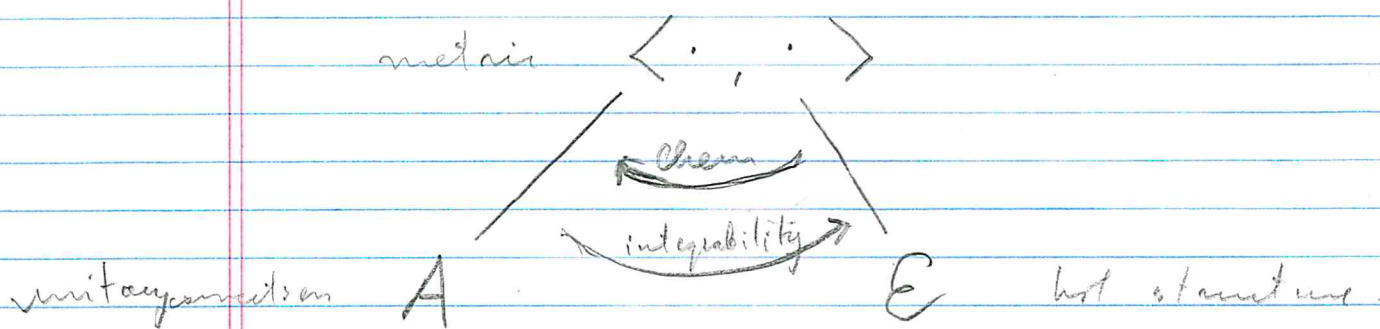
So for  $N$  sufficiently small, the matrix  $g = 1+f$  is invertible  $\in GL(n, \mathbb{C})$ .

$\therefore g$  solves (\*) on  $D_{1/2} \ni x_0$   $\square$ .

Note: if  $\dim M > 1$ , any unitary comm'n with  $F_A$  of type (1,1) defines a hol. structure.

(indeed:  $\left[ \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2} + a_2 \right] = 0 \Rightarrow \frac{\partial a_2}{\partial \bar{z}_1} = 0 \Rightarrow a_2(z_1, z_2)$  hol. in  $z_1$ )

# BASIC PICTURE:



Fix the  $C^\infty$  bundle  $E$  w/ unitary metric.

$$\text{unitary conn. } A \xleftrightarrow[\text{bijection}]{\quad} \bar{\partial}_E$$

Question

Two hol. structures  $\bar{\partial}, \bar{\partial}'$  on  $E$  are

isomorphic  $\Leftrightarrow \exists \quad E \xrightarrow{g} F$  hol. bundle isomorphism.

$\Rightarrow g \in T(E \otimes E^*)$  is invertible

and satisfies  $\bar{\partial}' g(s) = 0 \quad \forall s \text{ with } \bar{\partial}_E s = 0$   
 $\quad \quad \quad = g(\bar{\partial}_E s)$

$$\Leftrightarrow \bar{\partial}' = g \circ \bar{\partial}_E \circ g^{-1}$$

This defines an action of  $g \in \mathfrak{gl}(E) \subset T(E \otimes E^*)$ ,

*Write exercise to the effect that  $\bar{\partial}_{AB} g = 0$ !*

the complexified gauge group  $(\pi^{-1}(x) \cap \mathfrak{gl}^c = GL(n, \mathbb{C}))$

and  $E \simeq F \Leftrightarrow E$  and  $F$  are in the same orbit of  $\mathfrak{gl}^c$ .

How does  $g \in \mathcal{G}$  change the associated Chern connection?

$$\begin{aligned} E \rightsquigarrow A &\Rightarrow \bar{\partial}_B = g \bar{\partial}_A g^{-1} = \bar{\partial}_A - \bar{\partial}_A g \cdot g^{-1} \\ F \rightsquigarrow B &= \bar{\partial}_B^2 = \bar{\partial}_A^2 + (\bar{\partial}_A g \cdot g^{-1})^* \end{aligned}$$

(recall proof of Chern's Lemma.) (-a7)

Note: if  $u \in \mathcal{G} \subset \mathcal{G}^{\mathbb{C}}$

$\hookrightarrow$  ordinary  $U(n)$  gauge group

$$\begin{aligned} \text{then } u^* = u^{-1} &\Rightarrow (\bar{\partial}_A u u^{-1})^* = u \bar{\partial}_A (u^{-1}) \\ &= -u u^{-1} \bar{\partial}_A u \cdot u^{-1} \\ &= -\bar{\partial}_A u \cdot u^{-1} \end{aligned}$$

Question: Which isomorphism classes ( $= \mathcal{G}^{\mathbb{C}}$ -orbits) contain a compatible connection which is Yang-Mills?

Prop: assume  $L$  is a  $U(1)$  bundle w/ hol.

structure  $\bar{\partial}_A$ . Normalise  $Vol(M)=1$ . Then  $\exists g \in \mathcal{G}^{\mathbb{C}}$  with

$$*F_{g(A)} = -2\pi i \deg(L) \quad (\text{constant!})$$

in fact,  $\text{Yang-Mills}$

Proof: action of  $\mathcal{G}^{\mathbb{C}}$ :  $\bar{\partial}_{g(A)} = \bar{\partial}_A - \bar{\partial}_A g \cdot g^{-1}$

$$\begin{aligned} \text{assume } g &\equiv e^h \Rightarrow g(A)^{0,1} = A^{0,1} - \bar{\partial} h \\ g(A)^{1,0} &= A^{1,0} + (\bar{\partial} h)^* \end{aligned}$$

Note:  $\mathcal{G}^{\mathbb{C}} \subset L \otimes L^* \simeq \mathbb{C}$ !

$A = \text{trivial on } \mathcal{G}^{\mathbb{C}}$

$Re h = \text{part not along } u(1)$

$$F_g(A) = d(g(A)) = dA + d(-\bar{\partial}h + (\bar{\partial}h)^*)$$

$$= F_A - 2\bar{\partial}h + \underbrace{\bar{\partial}\bar{\partial}h^*}_{\text{cancel}} = -2\bar{\partial}h$$

$$F_g(A) - F_A = -2 \underbrace{\bar{\partial}\bar{\partial}h}_{\text{cancel}} d\bar{z} \wedge d\bar{z} = -2 \bar{\partial}\bar{\partial}h$$

$$*F_g(A) - *F_A = -2\Delta h$$

$$*F_g(A) + 2\pi i d = -2\Delta h + *F_A + 2\pi i d$$

Chern-Weil:  $\frac{i}{2\pi} \int X F_A dV = \frac{(-1)^2}{2\pi} \int T_n F_A = d$

$$\Rightarrow \int *F_A dV = -2\pi i d \cdot \text{Vol}(M)$$

So can solve  $RHS = 0$  !  $\square$

Note: also true on Kähler mfd's of

dimension  $> 1$  ( $\bar{\partial}\bar{\partial}$ -lemma!)

Cor: If  $\deg(L) = 0$  then  $L$  is (isotopically) isomorphic to a flat bundle.

Want to generalize to higher-rank bundles.

PDE already too difficult.  $(\bar{\partial}g)^2 = \text{nonlinear operator}$

Will try variational approach: minimize

$$YM(A) = \frac{1}{2} \|F_A\|^2 \text{ within a } \mathcal{G}^E\text{-orbit.}$$

For unitary,  
does  $*$  include  
 $Re$  - ?

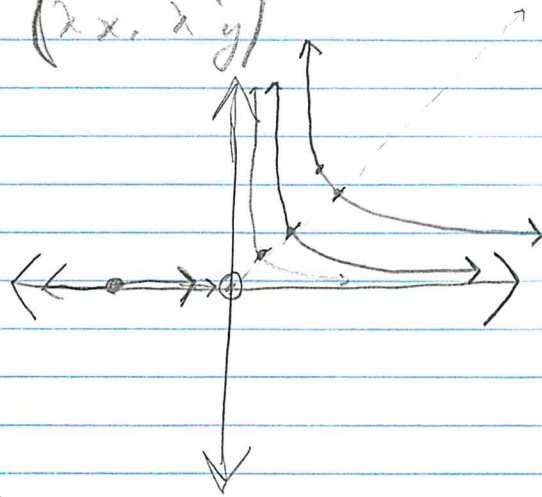
No, but  
includes an  
extra - !

due to  $\bar{\partial}\bar{\partial}$  lemma  
defn of  $*$ .

# Finite-dimensional analogy:

$$\mathbb{P}^1 \hookrightarrow \mathbb{C}^2$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (\lambda x, \lambda^{-1} y)$$



Closed orbits:

$$\{(\lambda x, \lambda^{-1} x) \mid x \neq 0\}$$

= "Stable orbits"

Non-closed orbits (and zero) = "Unstable orbits"

$$\{(x, 0) \mid x \neq 0\}, \{(0, y) \mid y \neq 0\}, \{0, 0\}$$

$$\text{let } |\bar{z}|^2 = |x|^2 + |y|^2$$

Observation: A nonzero orbit  $\mathcal{O}$  is unstable

$\Leftrightarrow |\bar{z}|^2$  attains a minimum in  $\mathcal{O}$

This minimum is unique up to  $S^1 \subset \mathbb{C}^\times$ .

Theorem (Kempf-Vess).

$\nearrow$  reductive group.

Let  $V$  be a  $\mathbb{C}$ -vector space with a  $\mathbb{C}^\times$ -action and Hermitian inner-product invariant under

$\rightarrow$  maximal cpt subgroup

$S' = U(1) \subset \mathbb{C}^*$ . Then an orbit  $\mathcal{O}$  is closed  $\Leftrightarrow \|\cdot\|$  attains a unique (up to  $U(1)$ ) minimum on  $\mathcal{O}$ .

Proof: Under  $\mathbb{C}^*$ ,  $V = \bigoplus_{i=1}^n \langle v_i \rangle$ , with  $\mathbb{C}^*$  acting by  $\lambda^{k_i}$  on  $v_i$ .

Let  $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n) \in V$ .

Inspection shows that  $\mathcal{O}(\bar{v})$  is stable  $\Leftrightarrow \bar{v}_i, \bar{v}_j \neq 0$  for some  $i, j$

with  $k_i > 0, k_j < 0$ .  $\square$

[ $\exists$  much deeper theory here comparing moduli when  $V \mapsto X$  alg. var.]  
Recall: We wish to find compact sets

in a  $g^c$ -orbit (isom. class) which minimize  $YM$ .

"Vector-space things go over to bundles"

$$\mathbb{C}^* \rightarrow g^c = \mathfrak{sl}(n, \mathbb{C}) \text{ twice}$$

$$U(1) \rightarrow g = \mathfrak{u}(n) \text{ twice.}$$

Stability  $\rightarrow ?$

Definition/Lemma (Mumford-Takemoto.)

Let  $\mu(E) = \deg(E) / \text{rk}(E)$  be

the slope of a  $\mathbb{C}$ -vector bundle  $E$ . [normalized Chern class]

a holomorphic structure  $\bar{\partial}$  on  $E$  is stable if for all proper holomorphic subbundles  $M \rightarrow E$

there holds  $\mu(m) \leq \mu(E)$ , or, equivalently  
 $\mu(E/m) \geq \mu(E)$ .

Proof of equivalence:  $\mu(E) = \frac{c_1(m) + c_1(E/m)}{rk(m) + rk(E/m)}$ .

Theorem (Maraimhan - Seshadri) Assume  $E$  is  
 indecomposable ( $E \neq m \oplus n$  for h.d. subbundles).

Then  $E$  is stable if and only if there is a compat.  
 unitary connection  $A$  with  

$$*F = -2\pi i \mu(E) \cdot Id.$$

Moreover,  $A$  is unique up to isomorphism.

Goal: Proof next time.

(also will review sub and quotients of h.d. bundles.)